An Introduction to Von Neumann Algebras

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Von Neumann algebras were introduced by Murray and von Neumann in a series of papers in the 1930s and 1940s, where the basic theory was developed. In older references, von Neumann algebras were often called W*-algebras. In this lecture, we see equivalent definitions of von Neumann algebra and some examples.

Notations :

$\mathcal{H}$		a complex Hilbert space with inner product $\langle .,. \rangle$.
$\mathcal{B}(\mathcal{H})$		the set of all bounded linear operators from $\mathcal{H}$ to itself.
$\mathbb{C}$		the set of complex numbers (the complex field).
$(\mathcal{A})_1$	$=$	$\{x \in \mathcal{A} : \|x\| \leq 1\}$ (the closed unit ball of $\mathcal{A}$).
$\mathcal{A}_{sa}$	$=$	$\{x \in \mathcal{A} : x^* = x\}$ (the set of self-adjoint operators in $\mathcal{A}$).

C*-algebra

Definition 1.

*A **(abstract) C*-algebra** is a Banach algebra $\mathcal{A}$ together with a maping $x \mapsto x^*$ on $\mathcal{A}$ satisfying the following conditions.*

(a) $(x^*)^* = x$ *for all* $x \in \mathcal{A}$.

(b) $(ax + by)^* = \bar{a}x^* + \bar{b}y^*$ *for all* $x, y \in \mathcal{A}$ *and* $a, b \in \mathbb{C}$.

(c) $(xy)^* = y^*x^*$ *for all* $x, y \in \mathcal{A}$.

(d) $\|x^*x\| = \|x\|^2$ *for all* $x \in \mathcal{A}$. *(C*-identity / C*-axiom).*

The element x^* is usually called the **adjoint of** x. Any mapping $x \mapsto x^*$ on an algebra satisfying (a), (b), and (c) is called an **involution** on the algebra.

We say $\mathcal{A}$ is **unital**, if there exists a unit element $1 \in \mathcal{A}$ ($1x = x1 = x$, for all $x \in \mathcal{A}$). Unit element is unique, if it exists.

C*-algebra

We point out that C*-identity is the most stringent condition in the definition of a C*- algebra.

- **The involution in a C*-algebra is an isometry :** In fact, for every x in $\mathcal{A}$ we have $\|x\|^2 = \|x^*x\| \leq \|x^*\| \, \|x\|$, so that $\|x\| \leq \|x^*\|$. Replacing x by x^* we conclude that $\|x^*\| \leq \|x\|$ and hence $\|x\| = \|x^*\|$.
- **Any involution $x \mapsto x^*$ satisfying $\|x\|^2 \leq \|x^*x\|$ must be an isometry, hence it satisfies C*-identity :** In fact, that this condition gives $\|x\|^2 \leq \|x^*x\| \leq \|x^*\| \, \|x\|$ and hence $\|x\| \leq \|x^*\|$. Using $x = (x^*)^*$, $\|x\| = \|x^*\|$ for all x in $\mathcal{A}$. Moreover $\|x\|^2 \leq \|x^*x\| \leq \|x^*\| \, \|x\| = \|x\|^2$, thus $\|x^*x\| = \|x\|^2$.
- **If $1 \in \mathcal{A}$, then by property $(xy)^* = y^*x^*$ implies that $1^* = 1$:** For every x in $\mathcal{A}$ we have $1^*x = (x^*1)^* = x, \quad x1^* = (1x^*)^* = x$. By the uniqueness of the unit in an algebra, $1^* = 1$.

Examples

The following are several examples of C*-algebras.

Example 2.

$\mathcal{A} = \mathbb{C}$ with $z^* = \bar{z}$ is the simplest C-algebra.*

Example 3.

For a compact Hausdorff space X, $\mathcal{A} = C(X)$ with $f^* = \bar{f}$ is a commutative C-algebra.*

Gelfand-Naimark Theorem says that every (unital) commutative C*-algebra is isomorphic to $C(X)$ for some compact Hausdorff space X. Note that if K consists of n distinct points, then $C(K) = \mathbb{C}^n$. Thus there exists a commutative C^*-algebra of any given dimension.

Examples

Example 4.

$\mathcal{A} = L^\infty(X, \mu)$ *with* $f^* = \bar{f}$ *is a commutative* C^**-algebra.*

Example 5.

$\mathcal{A} = \mathcal{B}(\mathcal{H})$ *with the usual adjoint operation as involution is a* C^**-algebra.*

If $\mathcal{H}$ *is n-dimensional, then* $\mathcal{B}(\mathcal{H}) = M_n(\mathbb{C})$ *is the algebra of all* $n \times n$ *complex matrices and the adjoint of a matrix in* $M_n(\mathbb{C})$ *is its conjugate transpose.*

C*-subalgebra

Definition 6.

Suppose $\mathcal{A}$ is a C-algebra and $\mathcal{B}$ is a closed (unital) subalgebra of $\mathcal{A}$ that is closed under the involution. Then $\mathcal{B}$ is a C^*-algebra by itself with the norm, involution, and algebraic structure inherited from $\mathcal{A}$. We call such $\mathcal{B}$ a* ***C*-subalgebra of*** *$\mathcal{A}$.*
In other words, C-subalgebra of $\mathcal{A}$ is a self-adjoint subalgebra $\mathcal{B}$ of $\mathcal{A}$ (closed under the involution), which is closed in the C^*-algebra $\mathcal{A}$.*

A subset of $\mathcal{B}(\mathcal{H})$ will be called self-adjoint if it contains the adjoint of any of its elements. A self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ will also be called a *-subalgebra. We shall show later that every C^*-algebra is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$. That is, every C*-algebra is a norm closed *-subalgebra of $\mathcal{B}(\mathcal{H})$ (for some $\mathcal{H}$). It is also true that every finite dimensional C^*-algebra is a C*-subalgebra of some $M_n(\mathbb{C})$.

Definition 7.

A ***(concrete) C*-algebra*** *is a self-adjoint subalgebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$, which is closed in the operator norm.*

As the operator norm satisfies the C*-identity, every concrete C*-algebra is an abstract C*-algebra.

Gelfand-Naimark-Segal Theorem says that every abstract C*-algebra can be expressed as a concrete C*-algebra.

We discuss the study of von Neumann algebras, a special class of C^*-subalgebras of $\mathcal{B}(\mathcal{H})$ (concrete C*-algebras).

Von Neumann algebras can be defined either algebraically or topologically. We begin with the weak- and strong- operator topologies on $\mathcal{B}(\mathcal{H})$.

In the sequel $\mathcal{H}$ will be a fixed Hilbert space.

Definition 8.

The ***strong-operator (or SO) topology*** *on* $\mathcal{B}(\mathcal{H})$ *is the locally convex topology generated by the semi-norms* $\|\ \|_x, x \in \mathcal{H}$ *(by the family of functions* $T \mapsto \|Tx\|$, x *ranges over* $\mathcal{H}$*), where*

$$\|T\|_x = \|Tx\|, \qquad T \in \mathcal{B}(\mathcal{H}).$$

The ***weak-operator (or WO) topology*** *on* $\mathcal{B}(\mathcal{H})$ *is the locally convex topology generated by the semi-norms* $\|\ \|_{x,y}, x, y \in \mathcal{H}$ *(by the family of complex valued functions on* $\mathcal{B}(\mathcal{H})$, $T \mapsto \langle Tx, y\rangle$, x *and* y *range over* $\mathcal{H}$*), where*

$$\|T\|_{x,y} = |\langle Tx, y\rangle|, \qquad T \in \mathcal{B}(\mathcal{H}).$$

SO and WO Topologies

Note that $T_\alpha \to T$ in the strong-operator topology if and only if

$$\|T_\alpha x - Tx\| \to 0, \qquad x \in \mathcal{H}.$$

And $T_\alpha \to T$ in the weak-operator topology if and only if

$$\langle T_\alpha x, y\rangle \to \langle Tx, y\rangle, \qquad x, y \in \mathcal{H}.$$

We have the following ordering of the topologies (strict in infinite dimensions) :

$$\text{weak topology} \subset \text{strong topology} \subset \text{norm topology}.$$

The weak topology is weaker than the strong topology, which in turn is weaker than the norm topology. Note that a weaker topology has less open sets so that a set is closed in the weak topology it is necessarily closed in the strong and norm topologies.

SO Topology : Basic Neighbourhoods

Let $\{T_\alpha\} \subseteq \mathcal{B}(\mathcal{H})$ be a net of bounded operators and let $T \in \mathcal{B}(\mathcal{H})$. We say that $\{T_\alpha\}$ converges strongly to T if

$$\lim_\alpha \| T_\alpha x - Tx\| = 0, \ \forall x \in \mathcal{H}.$$

The topology induced by this convergence is called the strong-operator topology (SOT).

The SOT is formed by basic neighbourhoods of the form

$$N(T, x_1, x_2, \ldots, x_n, \varepsilon) = \Big\{ S : \|(S - T)x_i\| < \varepsilon, \forall i = 1, 2, \ldots, n \Big\}.$$

WO Topology : Basic Neighbourhoods

Let $\{T_\alpha\} \subseteq \mathcal{B}(\mathcal{H})$ be a net of bounded operators and let $T \in \mathcal{B}(\mathcal{H})$. We say that $\{T_\alpha\}$ converges weakly to T if

$$\lim_{\alpha} |\langle (T_\alpha - T)x, y\rangle| = 0, \ \forall x, y \in \mathcal{H}.$$

The topology induced by this convergence is called the weak-operator topology (WOT).

The WOT is formed by basic neighbourhoods of the form

$$N(T, x_1, \ldots, x_n, y_1, \ldots, y_n, \varepsilon) = \Big\{ S : |\langle (S - T)x_i, y_i\rangle| < \varepsilon, \forall i = 1, 2, \ldots, n \Big\}.$$

By the Cauchy-Schwarz inequality, strong-operator convergence implies weak-operator convergence. When the dimension of $\mathcal{H}$ is infinite, the weak-operator topology is strictly weaker than the strong-operator topology.

SO and WO Continuous Functionals

A subset of $\mathcal{B}(\mathcal{H})$ will be called self-adjoint if it contains the adjoint of any of its elements. A self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ will also be called a *-subalgebra.

Theorem 9.

Suppose S is a convex subset of $\mathcal{B}(\mathcal{H})$. Then the WO closure of S coincides with the SO closure of S in $\mathcal{B}(\mathcal{H})$.

Definition 10.

*A C^*-subalgebra A of $\mathcal{B}(\mathcal{H})$ is called a **von Neumann algebra** if $\mathcal{A}$ is closed in the strong operator topology.*
In other words, a von Neumann algebra is a strongly closed (weakly closed) unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$.

We have discussed that a C*-algebra is a norm closed *-subalgebra of $\mathcal{B}(\mathcal{H})$ (for some $\mathcal{H}$). A von Neumann algebra is a weakly closed self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ (for some $\mathcal{H}$) containing the identity operator I.

By Theorem 9, a C^*-subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra if and only if $\mathcal{A}$ is called in the weak-operator topology. We shall see later that von Neumann algebras can also be defined in terms of the commutant, a purely algebraic notion.

There are several other topologies on the space of bounded operators, and one can ask what are the *-subalgebras closed in these topologies. If M is closed in the norm topology then it is a C*-subalgebra, but not necessarily a von Neumann algebra. One such example is the C*-subalgebra of compact operators (on an infinite dimensional Hilbert space).

For most other common topologies the closed *-subalgebras containing 1 are von Neumann algebras; this applies in particular to the weak operator, strong operator, *-strong operator, ultraweak, ultrastrong, and *-ultrastrong topologies.

Since the weak topology is weaker than the norm topology, a von Neumann algebra is, in particular, a C*-algebra. However, the "interesting" von Neumann algebras all turn out to be non-separable with respect to the norm topology. This should be taken as an indication that it would not be fruitful to regard von Neumann algebras simply as a particular kind of C*-algebras; rather, von Neumann algebras have their own, to some degree separate, theory. (This is not to suggest that no knowledge about C*-algebras will be useful when studying von Neumann algebras.)

Difference Between C*-algebras and Von Neumann Algebras

The difference between C*-algebras and von Neumann algebras is that C*-algebra theory is non-commutative topology (of locally compact spaces, presumably), while von Neumann algebra theory is non-commutative measure theory. This grows out of the observation that all commutative C*-algebras are isomorphic to the *-algebra $C_0(X)$ of all complex-valued continuous functions on some locally compact X (with the sup-norm), whereas all commutative von Neumann algebras are isomorphic to $L_\infty(Y, \mu)$ for some σ-finite measure space (Y, μ). The reader is encouraged to ponder what the locally compact space X would look if we wanted to realize the isomorphism $C(X) \cong L_\infty(Y, \mu)$, where $Y = \mathbb{N}$ with the counting measure, or even $Y = [0, 1]$ with Lebesgue measure. (This should help convince the skeptic why von Neumann algebras merit having their own special theory.)

Von Neumann Algebras - Examples

1. Let (X, μ) be a σ-finite standard measure space. Each $f \in L_\infty(X, \mu)$ gives rise to a bounded operator m_f on $L_2(X, \mu)$, defined by

$$(m_f(\phi))(x) = f(x)\phi(x).$$

   The set $\{m_f : f \in L_\infty(X, \mu)\}$ is an abelian von Neumann algebra, which may be seen to be a maximal abelian subalgebra of $\mathcal{B}(L_2(X, \mu))$. It can be shown that any abelian von Neumann algebra looks like this.

2. $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra. In particular, when $\mathcal{H}$ has finite dimension n, $\mathcal{B}(\mathcal{H})$ is the algebra $M_n(\mathbb{C})$ of $n \times n$ matrices over the complex numbers.

Von Neumann algebras are easy to come by. The full algebra $\mathcal{B}(\mathcal{H})$ is clearly a von Neumann algebra. If $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, then the weak- or strong-operator closure of $\mathcal{A}$ is a von Neumann algebra. To see that we can get nontrivial commutative von Neumann algebras this way, let T be a normal operator and let $\mathcal{A}$ be the C^*-subalgebra generated by T. It is clear that $\mathcal{A}$ is commutative. Since taking the weak-operator closure preserves commutativity, we see that the weak-operator closure of $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$ is a nontrivial commutative von Neumann algebra.

Theorem 11.

Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Then

1. $(\mathcal{A})_1$ *is compact in the WOT.*
2. $(\mathcal{A})_1$ *and* $\mathcal{A}_{sa}$ *are closed in the weak and strong topologies, so* $(\mathcal{A})_1$ *and* $\mathcal{A}_{sa}$ *are von Neumann algebras.*

PROOF :

1. Since the unit ball in $\mathcal{B}(\mathcal{H})$ is compact in WOT, $(\mathcal{A})_1$ is compact in the WOT.
2. Since taking adjoints is continuous in the WOT, it follows that $\mathcal{A}_{sa}$ is closed in the WOT, and by (1), this is also the closed for $(\mathcal{A})_1$.

In this lecture we show that von Neumann algebras can also be characterized in purely algebraic terms.

Definition 12.

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{F}$ be a subset of $\mathcal{B}(\mathcal{H})$. The set

$$\mathcal{F}' = \{T \in \mathcal{B}(\mathcal{H}) : TS = ST \text{ for all } S \in \mathcal{F}\}$$

is called the ***commutant*** *of $\mathcal{F}$. The* ***double commutant*** *of $\mathcal{F}$ is denoted by $\mathcal{F}''$.*

It is clear that $\mathcal{F}$ is always contained in $\mathcal{F}''$. For a subset $\mathcal{F}$ of $\mathcal{B}(\mathcal{H})$ we let $\mathcal{F}^* = \{T^* : T \in \mathcal{F}\}$. We say that $\mathcal{F}$ is **self-adjoint** if $\mathcal{F}^* = \mathcal{F}$.

The Commutant

Observed that, regardless of the structure of $\mathcal{F}$, $\mathcal{F}'$ is always a unital algebra. If $\mathcal{F}$ is closed under involution, then $\mathcal{F}'$ is a *-subalgebra.

Note that the inclusion are reversed under the commutant :

$$\mathcal{F}_1 \subseteq \mathcal{F}_2 \quad \text{implies} \quad \mathcal{F}_2' \subseteq \mathcal{F}_1'.$$

It is also check (algebracially) that

$$\mathcal{F} \subseteq \mathcal{F}'' = (\mathcal{F}'')'' = \cdots$$

$$\mathcal{F}' = (\mathcal{F}')'' = \cdots$$

Remarkably, the purely algebraic definition of the commutant has analytic implications. This culminates in the bicommutant theorem.

The Commutant

Note that (by separate continuity of composition), $\mathcal{F}'$ is both weakly and strongly closed. Furthermore, if $\mathcal{F}$ is self-adjoint, then so is $\mathcal{F}'$, and thus $\mathcal{F}'$ is a von Neumann algebra when $\mathcal{F}$ is self-adjoint (Proposition 13). In particular, the double commutant $\mathcal{F}'' = (\mathcal{F}')'$ is a von Neumann algebra containing $\mathcal{F}$.

Proposition 13.

$\mathcal{F}'$ is a von Neumann algebra if $\mathcal{F}$ is self-adjoint.

PROOF : $\mathcal{F}'$ is clearly a self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ containing the identity operator (Exercise). To see that $\mathcal{F}'$ is closed in the weak-operator topology, let $T_\alpha \to T$ with each $T_\alpha \in \mathcal{F}'$. For each S in $\mathcal{F}$ and x, y in $\mathcal{H}$ we have

$$\begin{aligned}\langle (TS - ST)x, y\rangle &= \langle TSx, y\rangle - \langle Tx, S^*y\rangle \\ &= \lim_\alpha \Big\{\langle T_\alpha Sx, y\rangle - \langle T_\alpha x, S^*y\rangle\Big\} \\ &= \lim_\alpha \langle (T_\alpha S - ST_\alpha)x, y\rangle = 0 \quad [\because T_\alpha \in \mathcal{F}'].\end{aligned}$$

It follows that $TS = ST$ and hence T is in $\mathcal{F}'$.

Invariant Subspaces

We recall the following results on invariant subspaces.

Proposition 14.

Suppose $\mathcal{H}$ is a Hilbert space and P is the orthogonal projection from $\mathcal{H}$ onto a closed subspace M. Then $PTP = TP$ if and only if M is an invariant subspace for T.

Proposition 15.

Suppose $\mathcal{H}$ is a Hilbert space and P is the orthogonal projection from $\mathcal{H}$ onto a closed subspace M. Then $TP = PT$ if and only if M is a reducing subspace for T.

Von Neumann's Double Commutant Theorem

Lemma 16.

Suppose $\mathcal{A}$ is a unital self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$. For each x in $\mathcal{H}$ and T in $\mathcal{A}''$ there exists a sequence $\{T_n\}$ in $\mathcal{A}$ such that $\|T_n x - Tx\| \to 0$.

PROOF : Let M be the closure of the subspace $\{Sx : S \in \mathcal{A}\}$ in $\mathcal{H}$. Since $\mathcal{A}$ is a self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$, M is a reducing subspace for each operator S in $\mathcal{A}$. Let P be the orthogonal projection form $\mathcal{H}$ onto M. Then by Proposition 14, $PS = SP$ for every $S \in \mathcal{A}$ and so $P \in \mathcal{A}'$. Since T is in $\mathcal{A}''$, we have $TP = PT$ and so M is a reducing subspace for T. In particular, $TM \subseteq M$. Since $\mathcal{A}$ is unital, the vector x ($x = 1x \in M$) belongs to M and so $Tx \in M$. This implies that there exists a sequence $\{T_n\}$ in $\mathcal{A}$ such that $\|T_n x - Tx\| \to 0$ because M is the closure of the subspace $\{Sx : S \in \mathcal{A}\}$ in $\mathcal{H}$.

Corollary 17.

Suppose $\mathcal{A}$ is a unital self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$. Given $x_1, \ldots, x_N$ in $\mathcal{H}$ and T in $\mathcal{A}''$ there exists a sequence $\{T_n\}$ in $\mathcal{A}$ such that

$$\lim_{n\to\infty} \|(T_n - T)x_k\| = 0, \qquad 1 \leq k \leq N.$$

PROOF : Let $\mathcal{H}_N = \mathcal{H} \oplus \cdots \oplus \mathcal{H}$($N$ copies). Define $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}_N)$ by $\Phi(T) = T \oplus \cdots \oplus T$ (N copies). It is easy to check that $\Phi(\mathcal{A})$ is a unital self-adjoint subalgebra of $\mathcal{B}(\mathcal{H}_N)$ and $\Phi(T) \in \Phi(\mathcal{A})''$. By Lemma 16, there exists a sequence $\{\Phi(T_n)\}$, with each T_n belonging to $\mathcal{A}$, such that

$$\|(\Phi(T_n) - \Phi(T))x\| \to 0, \qquad n \to \infty,$$

where $x = (x_1, \ldots, x_N)$ is in $\mathcal{H}_N$. It follows that $\|(T_n - T)x_k\| \to 0$, $n \to \infty$, for all $1 \leq k \leq N$.

We can now prove the main result of this lecture.

Theorem 18.

Suppose $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$. Then $\mathcal{A}$ is a von Neumann algebra if and only if $\mathcal{A} = \mathcal{A}''$.

PROOF : Since $\mathcal{A}$ is a C^*-subalgebra, $\mathcal{A}$ is self-adjoint and hence $\mathcal{A}'$ is self-adjoint (Exercise). By Proposition 13, $\mathcal{A}''$ is a von Neumann algebra. Since $\mathcal{A} = \mathcal{A}''$, $\mathcal{A}$ is a von Neumann algebra.

Suppose $\mathcal{A}$ is a von Neumann algebra, we proceed to show that $\mathcal{A} = \mathcal{A}''$. Since $\mathcal{A} \subset \mathcal{A}''$ and $\mathcal{A}$ is SO closed, it suffices to show that $\mathcal{A}$ is SO dense in $\mathcal{A}''$.

Given T in $\mathcal{A}''$ and a basic SO neighborhood of T of the form

$$\mathcal{U} = \{S \in \mathcal{B}(\mathcal{H}) : \|(S - T)x_k\| < \varepsilon, 1 \leq k \leq N\}.$$

By Corollary 17, there exists a sequence $\{T_n\}$ in $\mathcal{A}$ such that

$$\|(T_n - T)x_k\| \to 0, \qquad n \to \infty$$

for every $1 \leq k \leq N$. It is then clear that $T_n \in \mathcal{U}$ for n large enough. This shows that every SO neighborhood of T contains elements of $\mathcal{A}$. Since T is arbitrary, we see that $\mathcal{A}$ is SO dense in $\mathcal{A}''$.

Von Neumann "Density" or "Bicommutant" Theorem

The following result (due to von Neumann, 1929) is also called von Neumann's double commutant theorem; it is clearly equivalent to Theorem 18.

Theorem 19.

Suppose $\mathcal{A}$ is a unital self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$. Then $\mathcal{A}$ is strong-operator (as well as weak-operator) dense in $\mathcal{A}''$, that is, $\mathcal{A}''$ is the strong-operator (as well as weak-operator) closure of $\mathcal{A}$.

PROOF : This is really a consequence of the proof of von Neumann's double commutant theorem.

Corollary 20.

Suppose $\mathcal{F}$ is a self-adjoint subset of $\mathcal{B}(\mathcal{H})$. Then $\mathcal{F}''$ is the smallest von Neumann algebra containing $\mathcal{F}$.

PROOF : It is clear that $\mathcal{F}''$ is a von Neumann algebra containing $\mathcal{F}$. If $\mathcal{A}$ is another von Neumann algebra such that $\mathcal{F} \subset \mathcal{A}$, then $\mathcal{A}' \subset \mathcal{F}'$ and so $\mathcal{F}'' \subset \mathcal{A}''$. By von Neumann's double commutant theorem, $\mathcal{F}'' \subset \mathcal{A}'' = \mathcal{A}$, so that $\mathcal{F}''$ is the smallest von Neumann algebra containing $\mathcal{F}$.

Definition 21.

*If $\mathcal{F}$ is a von Neumann algebra, then $\mathcal{F}'$ is a von Neumann algebra. Consequently, $\mathcal{F} \cap \mathcal{F}'$ is a von Neumann algebra. This von Neumann subalgebra $\mathcal{F} \cap \mathcal{F}'$ is called the **centre** of $\mathcal{M}$, denoted by $Z(\mathcal{M})$.*

If $Z(\mathcal{M}) = \mathbb{C}1$, we say that $\mathcal{M}$ is a **factor**. If $Z(\mathcal{M}) = \mathcal{M}$, we say that $\mathcal{M}$ is **abelian**.

Commutative Von Neumann Algebra

Let $\mathcal{F}$ be a subset of $\mathcal{B}(\mathcal{H})$. Since the intersection of a family of von Neumann algebras is again a von Neumann algebra, there exists a smallest von Neumann algebra containing $\mathcal{F}$. We call this algebra the von Neumann algebra generated by $\mathcal{F}$. It is clearly the intersection of all von Neumann algebras containing $\mathcal{F}$. If $\mathcal{F}$ is self-adjoint, the corollary above shows that $\mathcal{F}''$ is the von Neumann algebra generated by $\mathcal{F}$. In particular, if $\mathcal{F} = \{T\}$ with $T^* = T$, then $\mathcal{F}''$ coincides with the strong-operator (as well as the weak-operator) closure of the set of polynomials in T. Similarly, if T is normal, then $\{T\}''$ is the strong-operator (and the weak-operator) closure of the set of polynomials in T and T^*. If $\mathcal{F}$ is self-adjoint and consists of mutually commuting operators, then $\mathcal{F}''$ is a commutative von Neumann algebra.

Commutative Von Neumann Algebra

The point here is that one uses the weak or strong operator topologies in an essential way in the study of von Neumann algberas. Moreover, there is no way to give an 'non-spatial' version of this defintion. A von Neumann algbera acts on a specific Hilbert space.

Example 22.

*Let (X, σ, μ) be a standard measure space. Let $L_2(X, \sigma, \mu)$ be the Hilbert space of square summable functions on X and let $L_\infty(X, \sigma, \mu)$ act as multiplication operators on this Hilbert space. That is, $\mathcal{A} = L_\infty(X, \sigma, \mu)$ is a *-subalgebra of $\mathcal{B}(L_2(X, \sigma, \mu))$. Then $\mathcal{A} = \mathcal{A}'$, that is, $\mathcal{A}$ is a maximal abelian and hence a von Neuman algebra. This is a commutative von Neumann algebra.*

Separable Von Neumann Algebra

The weak operator topology is weaker than the norm topology. Hence every von Neumann algebra is also a C*-algebra. One needs to be a little careful with that statement. For example, if one refers to a "separable von Neumann", it is usually assumed that this refers to the WO topology. Indeed, the only von Neumann algebras which are separable in the norm topology are finite-dimensional.

References

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